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Bounds for the connective constant of the hexagonal lattice

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Abstract
We give improved bounds for the connective constant of the hexagonal lattice. The lower bound is found by using Kesten's method of irreducible bridges and by determining generating functions for bridges on one-dimensional lattices. The upper bound is obtained as the largest eigenvalue of a certain transfer matrix. Using a relation between the hexagonal and the (3.12^2) lattices, we also give bounds for the connective constant of the latter lattice.

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1. Introduction

In this work we give improved bounds for the connective constant of the hexagonal lattice. However, the methods used are valid for a larger class of lattices.

The main motivation is to improve the partial order induced by strict bounds for connective constants for different lattices, studied in [3]. In order to separate the hexagonal lattice from the (4.8^2) lattice, we needed to apply a non-standard application of Kesten's method, which motivated a separate paper on the hexagonal lattice. A related partial order defined by percolation thresholds is studied in [10, 12].

1.1. Self-avoiding walks

A *walk* of length n on a lattice is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, e_2, \ldots, e_n, v_n\}$ such that the edge e_i connects the vertices v_{i-1} and v_i. The walk is *self-avoiding* if all vertices $v_0, v_1, \ldots, v_n$ are distinct.

For a regular graph, let f_n denote the number of self-avoiding walks, starting at a fixed vertex. Hammersley [6] proved that there exists a constant μ, called the *connective constant* of the lattice, such that

$$\lim_{n\to\infty} f_n^{1/n} = \mu.$$

Define the generating function for self-avoiding walks, sometimes called the *susceptibility*, by ($f_0 = 1$ by convention)

$$F(x) = \sum_{n=0}^{\infty} f_n x^n.$$

The generating function has radius of convergence $x_c = \mu^{-1}$.

The connective constant is unknown for all (truly) two-dimensional lattices, although Nienhuis [9] has presented strong evidence through non-rigorous computations that the connective constant for the hexagonal lattice equals $\sqrt{2+\sqrt{2}} \approx 1.847\,759$.

1.2. Bounds for the connective constant

Since the connective constants are unknown for most lattices, many papers concern bounds for the constants.

The first bounds for the connective constant, $\mu = \mu_{\text{hex}}$, of the hexagonal lattice were given by Fisher and Sykes [5] in 1959. They obtained the bounds

$$1.7872 < \mu < 1.9276$$

and also enumerated f_n, for $n \leqslant 20$. Sykes *et al* [11] extended the enumeration to $n \leqslant 34$ in 1972.

In his pioneering paper, Hammersley [6] showed that

$$\mu < f_n^{1/n}$$

which, using f_{34} gives

$$\mu < f_{34}^{1/34} < 1.9232$$

although Sykes *et al* did not give this bound in their paper.

Ahlberg and Janson [1] used the above enumeration, and the fact that for the hexagonal lattice

$$\mu < (f_n/f_2)^{1/(n-2)} \tag{1}$$

to show that

$$\mu < (f_{34}/f_2)^{1/32} < 1.895.$$

In 1993, Alm [2] showed that

$$\mu < (\lambda_1(G(m,n)))^{1/(n-m)} \tag{2}$$

where λ_1 denotes the largest eigenvalue of the matrix $G(m,n) = (g_{ij})_{K_m \times K_m}$, where the element g_{ij} is the number of n-stepped self-avoiding walks that start with γ_i and end with a translation of γ_j, and $\gamma_1, \ldots, \gamma_{K_m}$ are the different m-stepped self-avoiding walks (after taking symmetry considerations into account).

Using (2) with $n = 34$, $m = 12$, gave $K_{12} = 736$, and the bound

$$\mu < 1.876\,03$$

which is the currently best available upper bound for μ.

To summarize, we know that

$$1.7872 < \mu < 1.876\,03$$

which should be compared with the supposedly correct value given by Nienhuis [9],

$$\mu^{(N)} = \sqrt{2+\sqrt{2}} \approx 1.847\,759.$$

The bounds differ by -0.061 (-3.3%) and $+0.028$ ($+1.5\%$) from this value, so that there is more room for improvement of the lower bound.

2. Upper bounds

Improved upper bounds are obtained by the method of Alm [2]. The improvement of computers in the last ten years makes it possible to handle much larger matrices as well as much longer self-avoiding walks.

Using (2) with $n = 45, m = 17$, we get $K_{17} = 17\,700$, and the bound

$$\mu < 1.868\,832.$$

This reduces the difference between the upper bound and $\mu^{(N)} = \sqrt{2+\sqrt{2}}$ to +0.021 (+1.1%).

The computation took 944 CPU hours on a 1 GHz PC.

From the matrix $G(m, n)$, we can also compute f_n, thereby extending the enumeration of f_n to $n \leqslant 45$. This was further extended to $n \leqslant 48$ by direct computations; f_{48} requiring 928 CPU hours.

The values of $f_n, n \leqslant 48$ are given in table 1 together with upper bounds for μ for $m = 17$ and $34 \leqslant n \leqslant 45$. For comparison, we also include the bounds (1) for $m = 2$, i.e. $\bar{\mu}_2 = (f_n/f_2)^{1/(n-2)}$, which are the best bounds that can be obtained directly from the enumeration.

3. Lower bounds

3.1. The method of Kesten

In [8], Kesten presents a method for finding lower bounds for the connective constant, based on so called *irreducible bridges*.

Given a fixed embedding of the lattice in the plane, let the coordinates for a vertex v be denoted by $(v(x), v(y))$. A *bridge* of length n is a self-avoiding walk such that

$$v_0(y) < v_i(y) \leqslant v_n(y) \qquad \text{for} \quad i = 1, \ldots, n-1.$$

Denote the number of bridges of length n by b_n, and the generating function for bridges by ($b_0 = 1$)

$$B(x) = \sum_{n=0}^{\infty} b_n x^n.$$

An *irreducible bridge* is a bridge that cannot be decomposed into two bridges. Denote the number of irreducible bridges of length n by a_n, and the generating function for irreducible bridges by ($a_0 = 0$)

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Kesten proved that the connective constants for bridges and irreducible bridges are the same as for self-avoiding walks,

$$\lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} f_n^{1/n} = \mu.$$

Further, $A(x)$ and $B(x)$ are related by

$$B(x) = \frac{1}{1 - A(x)}.$$

Then, since $A(x)$ and $B(x)$ both have radius of convergence μ^{-1}, it follows [8, theorem 5] that the solution of $A(x) = 1$ is μ^{-1}.

Table 1. Number of self-avoiding walks, f_n, and upper bounds, $\bar{\mu}_m$, for $m = 2$ and $m = 17$.

n	f_n	$\bar{\mu}_2$	$\bar{\mu}_{17}$
1	3		
2	6		
3	12	2.000 000	
4	24	2.000 000	
5	48	2.000 000	
6	90	1.967 990	
7	174	1.961 010	
8	336	1.955 982	
9	648	1.952 042	
10	1218	1.942 840	
11	2328	1.939 314	
12	4416	1.935 031	
13	8388	1.931 770	
14	15 780	1.927 509	
15	29 892	1.924 934	
16	56 268	1.921 862	
17	106 200	1.919 545	
18	199 350	1.916 866	
19	375 504	1.914 895	
20	704 304	1.912 692	
21	1 323 996	1.910 950	
22	2 479 692	1.909 029	
23	4 654 464	1.907 493	
24	8 710 212	1.905 836	
25	16 328 220	1.904 467	
26	30 526 374	1.902 999	
27	57 161 568	1.901 771	
28	106 794 084	1.900 472	
29	199 788 408	1.899 364	
30	372 996 450	1.898 197	
31	697 217 994	1.897 191	
32	1 300 954 248	1.896 140	
33	2 430 053 136	1.895 223	
34	4 531 816 950	1.894 268	1.872 434
35	8 459 583 678	1.893 427	1.872 094
36	15 769 091 448	1.892 556	1.871 650
37	29 419 727 280	1.891 782	1.871 334
38	54 816 035 922	1.890 984	1.870 930
39	102 216 080 286	1.890 269	1.870 635
40	190 380 602 052	1.889 534	1.870 266
41	354 843 312 276	1.888 871	1.869 989
42	660 671 299 170	1.888 191	1.869 650
43	1 230 891 734 724	1.887 575	1.869 390
44	2 291 023 353 264	1.886 944	1.869 077
45	4 266 787 588 320	1.886 370	1.868 832
46	7 939 282 155 480	1.885 783	
47	14 780 995 214 220	1.885 246	
48	27 495 750 661 500	1.884 698	

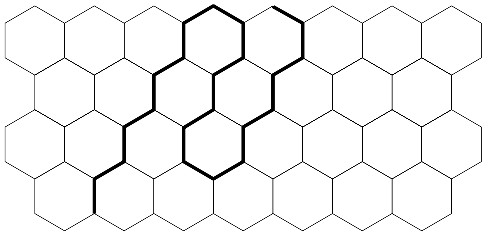

Figure 1. The hexagonal lattice, with an irreducible bridge of height 4.

Lower bounds for μ follow from the observation that, if $0 \leqslant \tilde{a}_n \leqslant a_n$, $n = 2, 3, \ldots, \infty$, then the reciprocal x_c^{-1} of the solution x_c to

$$\sum_{n=1}^{\infty} \tilde{a}_n x^n = 1$$

is a lower bound for the connective constant μ. Note that by letting $\tilde{a}_n = 0$ for $n > N$, this allows us to truncate the infinite series.

Consider the standard embedding of the hexagonal lattice, as shown in figure 1. Define the *height* of a walk from v_0 to v_n as the minimum number of vertical edges in any walk from v_0 to v_n. An irreducible bridge of length 22 and height 4 is shown in figure 1. Let b_n^N and a_n^N denote the number of bridges and irreducible bridges of length n and height N, and denote their generating functions by $B_N(x)$ and $A_N(x)$. Since $\sum_{k=1}^{\infty} a_n^k = a_n$, the reciprocal of the solution to

$$\sum_{k=1}^{N} A_k(x) = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{N} a_n^k \right) x^n = 1$$

is a lower bound for the connective constant μ.

Note that irreducible bridges must end at a top vertex of a hexagon, and that an irreducible bridge of height $N \geqslant 2$ must use at least three vertical edges of each height except the first, and have length at least $6N - 2$. For height 1, there are two irreducible bridges of each even length $n \geqslant 2$ (an irreducible bridge on the hexagonal lattice must have length at least 2).

Also, as every bridge either is an irreducible bridge, or can be decomposed into one irreducible bridge and one bridge, the following relation holds

$$B_N(x) = A_N(x) + A_{N-1}(x)B_1(x) + A_{N-2}(x)B_2(x) + \cdots + A_1(x)B_{N-1}(x).$$

The generating function $A_N(x)$ can thus be obtained from $A_k(x)$, $k = 1, \ldots, N-1$ and $B_k(x)$, $k = 1, \ldots, N$, by

$$A_N(x) = B_N(x) - A_1(x)B_{N-1}(x) - A_2(x)B_{N-2}(x) - \cdots - A_{N-1}(x)B_1(x). \qquad (3)$$

In the next section we will see that, using methods from Alm and Janson [4], it is theoretically possible to calculate $B_N(x)$, and thus $A_N(x)$, in finite time for any fixed height N.

3.2. *One-dimensional self-avoiding walks and bridges*

In [4], Alm and Janson have studied self-avoiding walks on one-dimensional lattices. For our purpose, it is sufficient to define a one-dimensional lattices as a horizontal strip of finite height of a two-dimensional lattice.

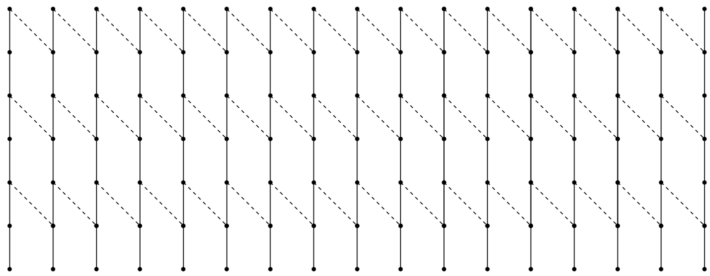

Figure 2. The embedding of the hexagonal lattice used for finding the generating function for bridges of height 3.

They show how to exactly compute the generating function for self-avoiding walks. The generating function is in principle obtained as one element in the inverse of a large matrix.

Only minor straightforward modifications are necessary to get the same result for the generating function for bridges.

To state the result precisely, we first need several definitions. Assume that the one-dimensional lattice is embedded in the plane in such a way that it consists of an alternating sequence of isomorphic *hinges* and isomorphic *sections*.

A hinge consists of vertices, all with the same x-coordinate, and all edges between these vertices. A section consists of all edges between two adjacent hinges. For the hexagonal lattice, we use embeddings of the type showed in figure 2, in which the hinges consist of the vertices and the vertical edges. The sections consist of the dotted edges. Note that in the graph in figure 2, self-avoiding walks that start at one of the bottom vertices, and ends at one of the top vertices, will be bridges of height 3, according to the usual embedding of the hexagonal lattice.

Consider a bridge on the lattice. The appearance of the bridge in a given section is called a *configuration*. The appearance of the bridge in a given hinge is called a *shape*. The shape of a particular hinge is not necessarily determined by the adjacent configurations, but a shape completely determines the configurations on both sides.

Let Σ be the set of all possible configurations, extended with two empty configurations, ϕ_L and ϕ_R. Any finite walk must by convention start with the configuration ϕ_L, and end with the configuration ϕ_R. Any finite bridge must start at the lowest level, and end at the highest level.

There is a one-to-one correspondence between bridges and *correctly connected* alternating sequences $\{\phi_L = c_0, s_1, c_1, \ldots, s_m, c_m = \phi_R\}$ of configurations and shapes, correctly connected meaning that for every shape s_i, c_{i-1} and c_i are the configurations determined by s_i.

The generating function $B_N(x)$ for bridges of some fixed height N is obtained from the following two square matrices, both indexed by Σ. $H = H(x)$ is a diagonal matrix with elements x^{h_i}, where h_i is the number of edges in configuration i. $V = V(x) = \{v_{ij}\}$ has elements $v_{ij} = \sum x^{v_k}$, where the sum is over all shapes k that can connect configuration i to the left, with configuration j to the right, and v_k is the number of edges in shape k. By convention, $v_{i\phi_L} = v_{\phi_R i} = 0$, for all i. Define the *generating matrix* $G = G(x)$ by $G = HV$.

Theorem 1 (Alm, Janson)**.**

$$B_N(x) = V_{\phi_L,\phi_R} + (VG)_{\phi_L,\phi_R} + (VG^2)_{\phi_L,\phi_R} + \cdots = (V(I-G)^{-1})_{\phi_L,\phi_R}.$$

Remark 1. A more straightforward way to get lower bounds from one-dimensional lattices would be to directly compute the connective constants for self-avoiding walks. However, this gives worse bounds than the method based on bridges, for one-dimensional lattices of the same heights.

3.3. Examples

We now illustrate theorem 1 with two examples, by calculating the generating function for irreducible bridges of heights 1 and 2.

For height 1, there are four possible configurations, $\Sigma = \{\phi_L, \nwarrow, \searrow, \phi_R\}$, with lengths 0, 1, 1, 0. The generating matrix G_1 is given by

$$G_1 = H_1 V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & x^2 & x^2 \\ 0 & x & 0 & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & x^2 & x^2 \\ 0 & x^2 & 0 & x^2 \\ 0 & 0 & x^2 & x^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generating function is given by

$$\begin{aligned} A_1(x) = B_1(x) = (V_1(I - G_1)^{-1})_{1,4} &= \frac{2x^2}{1 - x^2} \\ &= 2x^2 + 2x^4 + 2x^6 + 2x^8 + \cdots. \end{aligned}$$

For height 2, there are eight configurations,

$$\Sigma = \left\{ \phi_L\ ,\ \begin{matrix} \searrow \\ \nwarrow \end{matrix}\ ,\ \begin{matrix} \nwarrow \\ \end{matrix}\ ,\ \begin{matrix} \\ \nwarrow \end{matrix}\ ,\ \begin{matrix} \searrow \\ \end{matrix}\ ,\ \begin{matrix} \\ \searrow \end{matrix}\ ,\ \phi_R\ ,\ \begin{matrix} \nwarrow \\ \searrow \end{matrix} \right\}.$$

The matrices H_2 and V_2 are given by

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 & x^2 & 1 & x^2 & x^4 & x^2 & x^4 & x^2 \\ 0 & x^2 & 0 & x^2 & x^2 & x^2 & x^2 & x^2 \\ 0 & 0 & x & x & 0 & 0 & x^3 & x^3 \\ 0 & 0 & x^3 & x & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & x & x & x & x \\ 0 & 0 & 0 & 0 & x^3 & x & x^3 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 & x^2 \end{pmatrix}.$$

The generating function is

$$\begin{aligned} A_2(x) &= B_2(x) - A_1(x)B_1(x) = (V_2(I - G_2)^{-1})_{1,7} - A_1(x)^2 \\ &= \frac{2x^{10}(x^4 + 4x^2 + 4)}{(1 - x^2)^2(1 + x^2)(1 - x^2 - x^4 - x^8)} = 8x^{10} + 24x^{12} + 58x^{14} + \cdots. \end{aligned}$$

3.4. Summary

In this section we summarize the method used for the lower bound. Assume that we have calculated the generating functions B_k for bridges of height $k = 1, \ldots, N$. From these we can calculate A_k, by equation (3). A lower bound for the connective constant μ is achieved by the reciprocal x_c^{-1} of the solution x_c to the equation $\sum_{k=1}^{N} A_k(x) = 1$.

Note that, if we underestimate the functions B_k, we get underestimates of the functions A_k, and therefore, still get a lower bound. It is therefore permissible to truncate the infinite series B_k, for the cases where it is feasible to carry out the symbolic inversion of the matrix, and also to truncate the infinite series representation of the inverse matrix.

Table 2. Number of irreducible bridges.

	Height						
n	1	2	3	4	5	6	7
2	2						
4	2						
6	2						
8	2						
10	2	8					
12	2	24					
14	2	58					
16	2	116	40				
18	2	226	248				
20	2	418	956				
22	2	764	2932	232			
24	2	1368	8158	2208			
26	2	2438	21 194	11 908			
28	2	4312	52 768	48 924	1456		
30	2	7612	127 424	174 384	18 656		
32	2	13 398	301 336	567 066	130 180		
34	2	23 564	701 240	1 734 242	669 868	9584	
36	2	41 398	1 613 176	5077 228	2 896 786	154 112	
38	2	72 708	3 678 486	14 399 854	11 186 920	1 321 320	
40	2	127 646	8 332 878	39 863 556	39 960 038	8 176 732	65 136
42	2	224 070	18 781 132	108 330 740	134 910 038	41 577 438	1 258 416
44	2	393 274	42 167 658	290 120 946	436 772 298	185 506 724	12 797 828
46	2	690 222	94 393 440	768 000 442	1 369 384 066	754 670 432	92 575 300
48	2	1 211 320	210 817 326	2 014 040 116	4 287 469 968	2 868 092 528	541 473 298
50	2	2 125 800	469 995 164	5 241 551 508	12 553 608 776	10 351 754 668	2 740 477 924
52	2	3 730 590	1 046 346 578	13 555 798 878	37 039 148 380	35 894 677 308	12 497 716 690
54	2	6 546 818	2 326 934 596	34 876 762 888	107 870 357 502	120 587 726 960	52 714 512 594
56	2	11 488 942	5 170 378 690	89 344 535 622	310 799 090 686	394 963 366 374	209 362 196 128
58	2	20 161 784	11 480 731 734	228 047 517 858	887 494 845 612	1 267 269 921 544	793 008 736 988
60	2	35 381 548	25 479 343 670	580 303 521 948	2 515 218 274 794	3 998 003 947 894	2 891 762 035 512
62	2	62 090 392	56 523 233 522	1 472 856 568 902	7 082 773 418 692	12 437 565 678 140	10 224 629 512 728
64	2	108 961 132	125 349 924 212	3 729 993 423 200	19 835 888 844 516	38 241 846 805 060	35 247 308 133 632
66	2	191 213 568	277 913 356 354	9 428 372 543 664	55 290 614 198 678	116 425 397 228 820	118 980 531 760 554
68	2	335 556 516	616 036 743 246	23 793 707 134 584	153 488 979 691 246	351 481 068 641 732	394 632 167 359 492
70	2	588 860 754	1 365 319 252 836	59 962 719 844 118	424 578 943 200 246	1 053 470 052 659 232	1 289 668 249 938 810
72	2	1 033 378 688	3 025 567 446 724	150 930 254 650 748	1 170 815 299 517 686	3 137 874 062 518 550	4 162 060 662 237 216
74	2	1 813 453 306	6 704 031 069 242	379 504 185 427 756	3 219 812 806 690 196	9 295 979 825 494 524	13 288 671 828 545 886
76	2	3 182 388 814	14 853 570 233 752	953 369 709 263 106	8 833 331 985 285 592	27 409 160 402 323 884	42 039 183 397 649 906
78	2	5 584 703 188	32 907 775 865 780	2 393 102 745 030 608	24 181 849 819 174 446	80 479 402 119 427 078	131 938 075 461 499 130
80	2	9 800 471 012	72 902 886 657 418	6 002 886 315 196 280	66 073 638 249 791 486	235 435 386 138 206 528	411 227 122 483 745 614
82	2	17 198 627 838	161 500 519 239 580	15 048 570 170 389 500	180 230 973 427 403 650	686 491 758 206 462 152	1 273 995 129 812 876 972
84	2	30 181 488 004	357 758 261 638 838	37 705 022 115 819 458	490 874 221 247 694 464	1 995 853 645 048 476 200	3 925 967 882 491 042 342

3.5. Results

For heights up to 4, we were able to find exact expressions for the generating function for irreducible bridges. These are given in section 3.6. For heights 5, 6 and 7, we were also able to determine the generating matrix G, but were unable to carry out the matrix inversion. Instead we truncated the infinite series representation of the inverse, by an iterative method described in section 3.7.

The number of irreducible bridges for heights up to 7, and lengths up to 84 are given in table 2.

To improve the lower bound, we have also computed the number of irreducible bridges of heights 8, 9 and 10, for lengths up to 58,

$$A_8(x) = 452\,960x^{46} + 10\,207\,872x^{48} + 120\,066\,252x^{50} + 99\,411\,028x^{52} + 657\,310\,510x^{54} + 37\,244\,196\,500x^{56} + 188\,193\,918\,866x^{58} + O(x^{60})$$

$$A_9(x) = 3204\,784x^{52} + 82\,465\,376x^{54} + 1100\,970\,508x^{56} + 10\,265\,702\,604x^{58} + O(x^{60})$$

$$A_{10}(x) = 22\,982\,032x^{58} + O(x^{60}).$$

The best lower bound found in this work is 1.833 009 764. The difference between this bound and $\mu^{(N)} = \sqrt{2+\sqrt{2}}$ is −0.015 (−0.8%), slightly smaller than the difference for the upper bound.

Remark 2. The computations for $n = 58$ were distributed over a large number of computers. The total computation time was 22 400 CPU hours (1.6 GHz).

3.6. The generating functions

Below, we give the generating function for heights up to 4. D_N and N_N denotes the denominator and numerator of A_N.

$$A_1(x) = \frac{2x^2}{1-x^2}$$

$$A_2(x) = \frac{2x^{10}(x^4+4x^2+4)}{(1-x^2)^2(1+x^2)(1-x^2-x^4-x^8)}$$

$$\begin{aligned}D_3(x) = {}&(x^4-1)^2(x^8-2x^6+3x^4-3x^2+1)(x^{38}-5x^{36}+14x^{34}-27x^{32}+36x^{30}-34x^{28}\\&+19x^{26}-2x^{24}-6x^{22}+11x^{20}-18x^{18}+27x^{16}-23x^{14}\\&+9x^{12}-x^{10}+4x^6-8x^4+5x^2-1)\end{aligned}$$

$$\begin{aligned}N_3(x) = {}&-2(x^{38}+4x^{36}-20x^{34}+45x^{32}-57x^{30}-6x^{28}+127x^{26}-243x^{24}\\&+205x^{22}-22x^{20}-54x^{18}-21x^{16}+137x^{14}-2x^{12}\\&-209x^{10}+147x^8+38x^6-34x^4-36x^2+20)x^{16}\end{aligned}$$

$$\begin{aligned}D_4(x) = {}&(x^2+1)^9(x^5-x^4+x^2-x+1)^2(x^5+x^4-x^2-x-1)^2(x^4+1)^2(x^{40}-4x^{38}\\&+4x^{36}+5x^{34}-17x^{32}+23x^{30}-16x^{28}-4x^{26}+17x^{24}-15x^{22}+8x^{20}\\&+5x^{18}-14x^{16}+5x^{14}+2x^{12}-2x^{10}+4x^8-x^6-4x^4+1)^2(x^{37}-x^{36}\\&-4x^{35}+6x^{34}+6x^{33}-15x^{32}-4x^{31}+19x^{30}+4x^{29}-14x^{28}-12x^{27}\\&+13x^{26}+17x^{25}-19x^{24}-13x^{23}+21x^{22}+14x^{21}-19x^{20}-18x^{19}+17x^{18}\\&+10x^{17}-14x^{16}-6x^{15}+15x^{14}+8x^{13}-10x^{12}-5x^{11}+4x^{10}+4x^9-8x^8\\&-3x^7+5x^6+x^5-x^4-x^3+2x^2-1)^2(x^{37}+x^{36}-4x^{35}-6x^{34}+6x^{33}\\&+15x^{32}-4x^{31}-19x^{30}+4x^{29}+14x^{28}-12x^{27}-13x^{26}+17x^{25}+19x^{24}\\&-13x^{23}-21x^{22}+14x^{21}+19x^{20}-18x^{19}-17x^{18}+10x^{17}+14x^{16}-6x^{15}\\&-15x^{14}+8x^{13}+10x^{12}-5x^{11}-4x^{10}+4x^9+8x^8-3x^7-5x^6+x^5\\&+x^4-x^3-2x^2+1)^2(x-1)^{12}(x+1)^{12}(x^8-x^6+x^4+x^2-1)^2(x^8-x^6\\&-x^4+x^2+1)^2(x^{11}+x^{10}-x^9-2x^8+x^6-x^5+x^3+x^2-x-1)^2\\&\times(x^{11}-x^{10}-x^9+2x^8-x^6-x^5+x^3-x^2-x+1)^2\\&\times(x^3+x^2-1)^4(x^3-x^2+1)^4\end{aligned}$$

$$\begin{aligned}N_4(x) = {}&-2(x^{11}+x^{10}-x^9-2x^8+x^6-x^5+x^3+x^2-x-1)(x^{11}-x^{10}-x^9+2x^8\\&-x^6-x^5+x^3-x^2-x+1)(x^{37}+x^{36}-4x^{35}-6x^{34}+6x^{33}+15x^{32}\\&-4x^{31}-19x^{30}+4x^{29}+14x^{28}-12x^{27}-13x^{26}+17x^{25}+19x^{24}-13x^{23}\\&-21x^{22}+14x^{21}+19x^{20}-18x^{19}-17x^{18}+10x^{17}+14x^{16}-6x^{15}-15x^{14}\end{aligned}$$

$$
\begin{aligned}
&+8x^{13}+10x^{12}-5x^{11}-4x^{10}+4x^{9}+8x^{8}-3x^{7}-5x^{6}+x^{5}+x^{4}-x^{3}\\
&-2x^{2}+1)(x^{37}-x^{36}-4x^{35}+6x^{34}+6x^{33}-15x^{32}-4x^{31}+19x^{30}+4x^{29}\\
&-14x^{28}-12x^{27}+13x^{26}+17x^{25}-19x^{24}-13x^{23}+21x^{22}+14x^{21}-19x^{20}\\
&-18x^{19}+17x^{18}+10x^{17}-14x^{16}-6x^{15}+15x^{14}+8x^{13}-10x^{12}-5x^{11}\\
&+4x^{10}+4x^{9}-8x^{8}-3x^{7}+5x^{6}+x^{5}-x^{4}-x^{3}+2x^{2}-1)(x^{198}-5x^{196}\\
&-94x^{194}+1439x^{192}-10\,077x^{190}+45\,706x^{188}-146\,052x^{186}+322\,139x^{184}\\
&-375\,099x^{182}-460\,084x^{180}+3735\,706x^{178}-10\,989\,654x^{176}\\
&+21\,041\,143x^{174}-26\,443\,197x^{172}+13\,762\,470x^{170}+26\,655\,182x^{168}\\
&-83\,450\,197x^{166}+116\,277\,400x^{164}-79\,099\,657x^{162}-26\,911\,408x^{160}\\
&+117\,139\,658x^{158}-66\,719\,812x^{156}-163\,339\,763x^{154}+420\,127\,625x^{152}\\
&-414\,373\,898x^{150}-40\,794\,436x^{148}+792\,055\,755x^{146}-1369\,812\,592x^{144}\\
&+1339\,768\,016x^{142}-713\,686\,008x^{140}+13\,379\,549x^{138}+149\,203\,153x^{136}\\
&+366\,222\,831x^{134}-1070\,571\,661x^{132}+1286\,138\,525x^{130}-776\,725\,830x^{128}\\
&-56\,615\,258x^{126}+595\,347\,458x^{124}-586\,136\,637x^{122}+301\,833\,490x^{120}\\
&-174\,524\,358x^{118}+329\,622\,043x^{116}-497\,688\,054x^{114}+351\,910\,601x^{112}\\
&+110\,012\,919x^{110}-541\,133\,599x^{108}+598\,537\,662x^{106}-284\,947\,505x^{104}\\
&-81\,751\,322x^{102}+193\,056\,310x^{100}-23\,672\,554x^{98}-188\,554\,083x^{96}\\
&+218\,784\,743x^{94}-61\,217\,653x^{92}-131\,592\,311x^{90}+207\,344\,673x^{88}\\
&-139\,738\,148x^{86}+41\,821\,218x^{84}-532\,196x^{82}+1689\,902x^{80}\\
&-25\,387\,860x^{78}+36\,161\,567x^{76}+1501\,343x^{74}-30\,406\,174x^{72}\\
&+12\,099\,108x^{70}+5503\,781x^{68}-11\,955\,348x^{66}+12\,806\,700x^{64}\\
&+5866\,625x^{62}-13\,834\,418x^{60}+1869\,202x^{58}+1479\,416x^{56}\\
&-6953\,954x^{54}+12\,983\,920x^{52}-1064\,824x^{50}-5192\,613x^{48}\\
&+61\,083x^{46}-2124\,568x^{44}+471\,817x^{42}+4762\,797x^{40}-9273x^{38}\\
&-2486\,033x^{36}-1428\,960x^{34}+153\,688x^{32}+1762\,191x^{30}+549\,830x^{28}\\
&-877\,409x^{26}-560\,376x^{24}+220\,009x^{22}+275\,271x^{20}+54\,321x^{18}\\
&-115\,656x^{16}-49\,531x^{14}+30\,788x^{12}+19\,929x^{10}-7606x^{8}-3448x^{6}\\
&+974x^{4}+404x^{2}-116)(x^{3}-x^{2}+1)^{2}(x^{3}+x^{2}-1)^{2}\\
&\times(x^{5}-x^{4}+x^{2}-x+1)^{2}(x^{5}+x^{4}-x^{2}-x-1)^{2}(x^{4}+1)^{2}\\
&\times(x^{8}-x^{6}-x^{4}+x^{2}+1)^{2}(x^{2}+1)^{5}(x-1)^{8}(x+1)^{8}x^{22}.
\end{aligned}
$$

3.7. Algorithmic issues

We have calculated the generating matrix for bridges of heights up to 7. The computations were done by a computer program written in C++. The program iterates through all possible pairs of configurations, finds all possible shapes that connects the two configurations and generates the matrix in Maple format.

The program is quite time consuming, the case of height 7 required a total computation time of several weeks of CPU time on a standard desktop computer. However, the matrix was divided into submatrices, which were generated on separate computers.

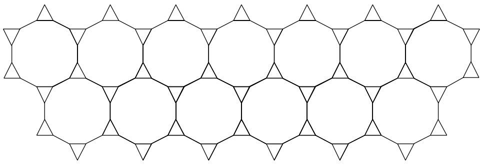

Figure 3. The (3.12^2) lattice.

Table 3. Summary of data for the computations of the generating functions.

Height	1	2	3	4	5	6	7
Dimension of generating matrix	4	8	18	44	118	338	1024
Iteration steps	∞	∞	∞	∞	130	100	42
Exact enumeration up to term	∞	∞	∞	∞	260	200	84

The matrices and generating functions were treated in Maple. For heights up to 4 Maple was able to handle the symbolic inversions. For heights 5, 6 and 7 we calculated a truncation of the infinite series representation, by the following iterative method.

Let J_{ϕ_R} be the column vector with a one in position ϕ_R, and zeros elsewhere, and let G_{ϕ_R} be column ϕ_R in G. Let $F^{(1)} = G_{\phi_R} + J_{\phi_R}$, and iteratively, for $k \geqslant 2$, $F^{(k)} = GF^{(k-1)} + J_{\phi_R}$. Then

$$(VF^{(n)})_{\phi_L} = V_{\phi_L,\phi_R} + (VG)_{\phi_L,\phi_R} + (VG^2)_{\phi_L,\phi_R} + \cdots + (VG^n)_{\phi_L,\phi_R}.$$

For heights 5, 6 and 7, we performed 130, 100 and 42 iteration steps. For height 7, we are currently unable to do further iterations, due to local computer limitations. In table 3, the dimensions of the generating matrices, and the number of iteration steps are summarized.

It is easy to check that n iteration steps give the correct number of bridges, and thus also irreducible bridges, for lengths up to $2n$. The lower bound is further improved by including all positive higher order terms.

4. Bounds for the (3.12^2) lattice

Jensen and Guttmann [7] show, by studying the generating functions for self-avoiding polygons, that the connective constant of the (3.12^2) lattice, see figure 3, is related to the connective constant of the hexagonal lattice through

$$\frac{1}{\mu_{\text{hex}}} = \frac{1}{\mu^2_{(3.12^2)}} + \frac{1}{\mu^3_{(3.12^2)}}. \tag{4}$$

Using Nienhuis' value [9], $\mu^{(N)} = \sqrt{2+\sqrt{2}}$ for the hexagonal lattice gives

$$\mu^{(N)}_{(3.12^2)} = \frac{12^{\frac{1}{3}} \cdot (2+\sqrt{2})^{\frac{1}{6}} \cdot \left(9 + \sqrt{81 - 12\sqrt{2+\sqrt{2}}}\right)^{\frac{1}{3}}}{6} + \frac{12^{\frac{1}{3}} \cdot (2+\sqrt{2})^{\frac{1}{6}} \cdot \left(9 - \sqrt{81 - 12\sqrt{2+\sqrt{2}}}\right)^{\frac{1}{3}}}{6} \approx 1.711\,041.$$

Using the relation (4) and the bounds for μ_{hex} from the previous sections, we get corresponding bounds for $\mu_{(3.12^2)}$

$$1.705\,263 < \mu_{(3.12^2)} < 1.719\,254.$$

Here, the bounds differ from the presumably correct value $\mu^{(N)}_{(3.12^2)}$ by -0.0058 (-0.34%) and +0.0082 (+0.48%), respectively.

Remark 3. The upper bound for $\mu_{(3.12^2)}$ obtained above is much better than the corresponding upper bound obtained by direct computations on the (3.12^2) lattice. Using (2) with $n = 48$ and $m = 18$ ($K_{18} = 23\,976$) gave, after 620 CPU hours, the upper bound

$$\mu_{(3.12^2)} < 1.729\,22$$

overestimating $\mu^{(N)}_{(3.12^2)}$ by 0.018 (1.1%).

Remark 4. Jensen and Guttmann [7] show that the generating function for self-avoiding polygons on the (3.12^2) lattice can be obtained from the corresponding generating function for the hexagonal lattice through the transformation $x \to x^2 + x^3$ (apart from an initial term $x^3/3$ corresponding to the triangles of (3.12^2)). This suffices to show the relation (4), as self-avoiding polygons have the same connective constant as self-avoiding walks.

However, Jensen and Guttmann also claim a similar relation between the generating functions for self-avoiding walks on these lattices, but their relation is incorrect, and it is doubtful whether such a relation exists.

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